Eulerian Numbers

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Definition (Permutation)

A permutation of [n] is a bijective function, σ, from $[n] \to [n]$. To avoid clutter it is useful to write a permutation in the so called *one line* notation, that is $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. So an example of a permutation of [3] is $\sigma = 213$, i.e. the permutation that maps $1 \to 2$, $2 \to 1$ and $3 \to 3$.

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The set of all permutations of $[n]$ is denoted by S_n.

What are Eulerian numbers?

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Given a permutation, $\sigma \in S_n$ an index i is said to be a *descent*, if $\sigma(i) > \sigma(i+1)$.

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Example

Let us compute $\left\langle {3 \atop 1} \right\rangle$, the permutations of [3] are:

$$123, 321, 213, 312, 231, 132$$

Of these the ones with 1 descent are $213, 312, 231, 132$, therefore $\left\langle {3 \atop 1} \right\rangle = 4$.

What are Eulerian Numbers?

Table: $\left\langle {n \atop k} \right\rangle$ **values for** $n \leq 10$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

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Several interesting mathematical facts about Eulerian numbers are hiding inside of this table.

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- The sequence of numbers in each row is unimodal.
- The sequence of numbers in each row is palindromic.
- The sum of the numbers in the nth row is $n!$.

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Theorem

For any $n > 0$ *and all* k, $\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$.

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Proof.

Given a permutation in S_n with k descents, we can delete the number n from the one line notation to obtain a permutation in S_{n-1} with k or $k-1$ descents. Conversely, given a permutation in S_{n-1} with k descents we can add n in $k+1$ positions to maintain k descents and given a permutation in S_{n-1} with $k-1$ descents we can add n to $n-k$ positions to get k descents, this gives us the desired result. □

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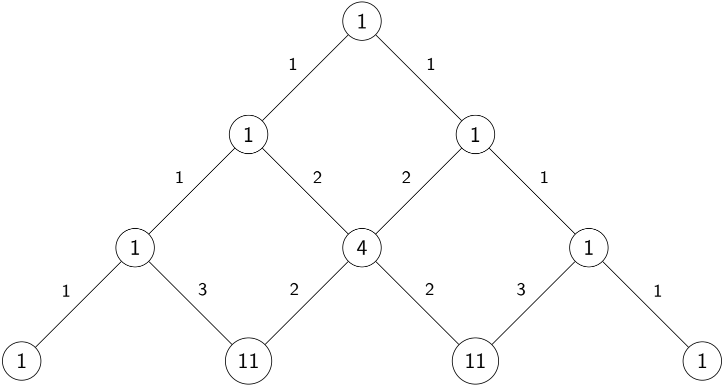

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To prove Worpitzky's identity bijectively we need the following definition:

Definition

A *barred* permutation is a permutation of $[n]$ with precisely k inserted bars, with the restriction that at least one bar must be inserted between a descent. We shall let $B(n,k)$ denote the number of barred permutations of $[n]$ with k bars. For example $B(3,2) = \mathrm{Card}(\{||123, |3|12, 3|2|1, ...\})$.

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Finally we are ready to give the proof of Worpitzky's identity:

Proof.

Let us count $B(n, k)$. We can obtain a barred permutation of $[n]$ with k bars from a normal permutation with i descents by placing a bar between each descent and then placing the remaining $k - i$ bars, the total number of ways to do this is $\left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}$ therefore $B(n, k) = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}$, but we can also count $B(n, k)$ in a different way, since the numbers between any two bars are increasing we have that $B(n, k)$ is equal to the number of partitions of the set $[n]$ into at most $k + 1$ ordered parts, this value is $(k + 1)^n$, equating the two obtained values we get the desired result.

□

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Theorem (Alternating sum formula)

For any $n \geq 1$ and all k, $\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i (k+1-i)^n \binom{n+1}{i}$

Eulerian Polynomials

Definition

An *Eulerian Polynomial* is a polynomial, $A_n(t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$. For convenience we define $A_0(t) = 1$(unfortunately this conflicts with the usual empty sum convention).

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$$A_{n+1}(t) = (1 + nt)A_n(t) + t(1 - t)A_n'(t)$$

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Continuing this process of differentiating we get:

$$\sum_{k=0}^{\infty} k^3x^k = \frac{x(1+4x+1x^2)}{(1-x)^4}$$

$$\sum_{k=0}^{\infty} k^4x^k = \frac{x(1+11x+11x^2+1x^3)}{(1-x)^5}$$

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This is known as the *Carlitz identity* and can be shown using induction and the recurrence for Eulerian polynomials.

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Where the B_k are the Bernoulli numbers and k is a non-negative integer.

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Where the B_k are the Bernoulli numbers and k is a non-negative integer. We can also evaluate $\zeta_a(-k)$, using the Carlitz identity. Letting $x = -1$ in the Carlitz identity, we get that $\zeta_a(-k) = \frac{-A(-1)}{2^{k+1}}$, solving for $A(-1)$ we get that $A(-1) = 2^{n+1}(2^{n+1} - 1)\frac{B_{n+1}}{n+1}$, in other words $\sum_{m=0}^{n}(-1)^m\left\langle {n \atop m} \right\rangle = (2^{n+1} - 1)\frac{B_{n+1}}{n+1}$, which gives us a relation between the Eulerian numbers and the Bernoulli numbers.

Thank You! Questions?