Artin's Primitive Root Conjecture

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Euler Circle

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Primitive Roots mod p

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So if we have a prime p, for which the decimal expansion of $\frac{1}{p}$ has period $p - 1$, the maximum possible, then $p - 1$ must the least positive k for which $10^k = 1 \mod p$ holds. In such a case, we say 10 is a primitive root mod p.

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An integer a is a primitive root mod p if the subgroup generated by a in the cyclic group $(\mathbb{Z}/p\mathbb{Z})^\times$ is the whole group.

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Quantitative Form

Given a non-zero integer a other than -1 or a perfect square, if $\mathcal{P}_a(x)$ denotes the number of primes less than equal to x for which a is a primitive root, then we have that $\mathcal{P}_a(x) \sim \delta(a)\frac{x}{\log x}$, where $\delta(a)$ is a specific positive function of a.

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In the qualitative form, $\delta(a)$ is the density or proportion of primes for which a is a primitive root since by the Prime Number Theorem $\pi(x) \sim \frac{x}{\log x}$. Of course, the quantitative form implies the qualitative form.

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Given a prime p, a is a primitive root mod p if and only if p does not split completely in any K_q, where q is prime and $K_q = \mathbb{Q}(\zeta_q, a^{1/q})$.

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Now, Chebotarev's Density Theorem implies that the density of primes which split in K_k is $\frac{1}{n(k)}$, where $n(k)$ is the degree of the extension $K_k/\mathbb{Q}$.

The function $\delta(a)$

Using Chebotarev's Density Theorem and the fact that a prime p splits completely in K_k and K_l if and only if it splits completely in $K_{\text{lcm}(k,l)}$, we can find a heuristic for $\delta(a)$ using the inclusion-exclusion principle:

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$\delta(a)$ gives us the density of primes which split in none of the K_q, for prime q. To "compute" this density subtract the density for each prime:

$$1 - \frac{1}{n(2)} - \frac{1}{n(3)} - \frac{1}{n(3)} - \dots$$

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And so on. In this way, we get that $\delta(a)$ "=" $\sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}$, where μ is the Möbius function.

The function $\delta(a)$

In the previous slide, we gave a heuristic for $\delta(a) = \sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}$. Letting a_1 be the square free part of a and h be the largest integer such that a is an h-th power, it turns out we have the following theorem:

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Theorem

Let $A(h) = \prod_{q \nmid h} \left(1 - \frac{1}{q(q-1)}\right) \prod_{q \mid h} \left(1 - \frac{1}{q-1}\right)$, *where q is prime. Then we have that*

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)} = \begin{cases} A(h) & \text{if } a_1 \neq 1 \mod 4 \\ \left(1 - \mu(|a_1|) \prod_{q \mid a_1, q \mid h} \frac{1}{q-2} \prod_{q \mid a_1, q \nmid h} \frac{1}{q^2-q-1}\right) A(h) & \text{if } a_1 = 1 \mod 4 \end{cases}$$

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Since $\sum_{k=1}^{\infty} \frac{1}{k(k-1)}$ converges, $A(h)$ is positive. Therefore, if the heuristic holds, then $\delta(a)$ is also positive and Artin's Primitive Root Conjecture is true.

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Hooley's Conditional Proof

Subject to the truth of the Generalized Riemann Hypothesis Cristopher Hooley proved that our heuristic value for $\delta(a)$ is indeed correct.

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Intuitively, the Generalized Riemann Hypothesis gives us an effective version of Chebatorev's Density Theorem, which allows us to make the inclusion-exclusion argument rigorous. Except not quite since the error term ends up being two large. Nevertheless, Hooley was able to introduce some intermediate quantities that made everything work.

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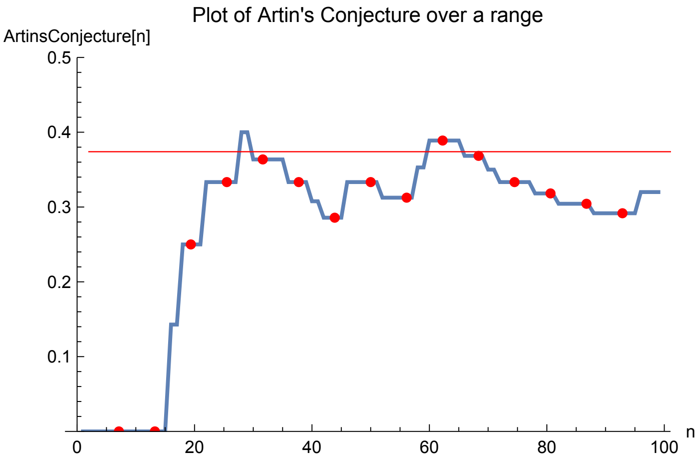
Hooley proved:

Theorem

$$\mathcal{P}_a(x) = \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{n(k)}\right) \frac{x}{\log x} + O\left(\frac{x \log \log x}{\log^2 x}\right)$$

Thank You! Questions?

Artin's Conjecture for $a = 10$



Credit: Navye Anand